Math 3435 Midterm

1. (10 points) Solve $(1 + x^2)u_x + u_y = 0$ with $u(0, y) = y$.

Solution: The equation is $((1 + x^2), 1) \cdot \nabla u = 0$, so u is constant on curves with $\frac{dy}{dx} = (1+x^2)^{-1}$. These curves are given by $y = \arctan(x)+c$, so $u(x, y) = f(c) = f(y-\arctan(x))$. Now $y = u(0, y) = f(y)$, so $u(x, y) = y - \arctan(x)$.

2. (10 points) Let $u(x,t)$ be the temperature on a long rod which has one end held at constant temperature zero (so $u = 0$ at this point). Treating the rod as a half-line $(0,\infty)$ solve the heat equation for the initial heat distribution

$$u(x,0) = \begin{cases} 1 & \text{if } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

Write your solution in terms of the error function at suitable points depending on x and t.

Solution: We have the heat equation with Dirichlet boundary data, so the solution is done by odd reflection as follows. Solve the problem for heat on $\mathbb{R}$ assuming that $u(-x,t) = -u(x,t)$ for all x and $t > 0$. Note that this makes the initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x \in [1,2] \\ -1 & \text{if } x \in [-2,-1] \\ 0 & \text{otherwise} \end{cases}$$

The solution to the heat equation on $\mathbb{R}$, which we consider only for $x > 0$ to solve our problem, is

$$\begin{aligned} u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)u(y,0)\,dy &= -\int_{-2}^{-1} S(x-y,t)\,dy + \int_{1}^{2} S(x-y,t)\,dy \\ &= \frac{1}{\sqrt{\pi}} \int_{(x+2)/\sqrt{4kt}}^{(x+1)/\sqrt{4kt}} e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}} \int_{(x-1)/\sqrt{4kt}}^{(x-2)/\sqrt{4kt}} e^{-p^2}\,(-dp) \\ &= \frac{1}{2}\Big(\mathrm{Erf}\big(\frac{x+1}{\sqrt{4kt}}\big) - \mathrm{Erf}\big(\frac{x+2}{\sqrt{4kt}}\big) + \mathrm{Erf}\big(\frac{x-1}{\sqrt{4kt}}\big) - \mathrm{Erf}\big(\frac{x-2}{\sqrt{4kt}}\big)\Big) \end{aligned}$$

where we made the change of variables $p = x - y$.

3. (10 points) State whether the following are true or false and give a one sentence explanation for each. (More than half the points are for the explanation.)

   (a) If u solves the heat equation on $[0, 1]$ and $u(0, t) = u(1, t) = 0$ then $M(t) = \max_{x\in[0,1]} u(x, t)$ has $\frac{dM}{dt} \leq 0$.

   (b) Solutions of the equation $4u_{xx} + 6u_{xt} + 2u_{tt} = 0$ which are zero outside a bounded set when $t = 0$ will be zero outside a bounded set for all times.

Solution:

(a) True. By considering $u(x, t - t_1)$ the zero boundary data and the maximum principle imply that the maximum on $[0, 1] \times [t_1, t_2]$ occurs on the boundary $t = t_1$

(b) True. Calculating the discriminant we find it is $3^2 - 4(2) = 1 > 0$ so the equation is hyperbolic; a change of variables makes it into a wave equation which will have finite propagation speed, which implies the statement.

4. (20 points) Let $u(x,t)$ describe the displacement of a long, taut string which has one end fixed. A measuring device at distance $2d$ from the fixed end detects the displacement of the string at this point.

   Suppose that at time $t = 0$ the string is plucked so as to produce an initial wave which is triangular, centered at d, and of height $2a$ and width $2a$. The initial velocity is zero and $0 < a < d$.

   Draw a graph of the displacement measured by the measuring device, labelling the maximum and minimum displacements and when they occur.

Solution: The problem is the wave equation on $(0,\infty)$ with Dirichlet boundary condition $u(0,t) = 0$ (a long string with one end fixed). You are asked to describe $u(2d,t)$, and are given $u(x,0)$. You could write an equation for it or draw a picture. I think the picture explanation is easier, and have written an explanation for it. Either solution is ok.

The initial data is a triangle shape of height a and base length $2a$ centered at d. The fact that $a < d$ ensures this shape fits in the space to the right of 0. Now the wave equation solution is two waves of half this height but the same shape, one moving left at speed c and the other moving right at speed c. The right-moving one will hit $2d$ first, at time $(d-a)/c$. The left-moving one will go left until it hits the fixed point and reflect with reversed sign so as to move right but with negative displacement, delayed a time $2d/c$ behind the right-moving wave, so it will hit $2d$ at time $(3d-a)/c$. It is then interesting to see if the right and left-moving waves overlap. They both have width $2a$, so go a units left and right of their centers, and the centers are $2d$ apart because the peak of the left-moving wave hits 0 at the same time as the right-moving wave hits $2d$. Since $2a < 2d$ they do not overlap. Accordingly the motion at $2d$ is that it has zero displacement until $t = (d-a)/c$, rises linearly until the peak of the right-moving wave hits at time d/c at which point it reaches its maximum displacement a, then it falls linearly until it his zero at $t = (d+a)/c$ and stays there until the wave that was originally left-moving hits at time $t = (3d-a)/c$, at which point it displaces linearly downward to a minimum height of $-a$ at time $3d/c$ and linearly back up to the neutral position at time $t = (3d+a)/c$. Thereafter it stays at neutral position.

To do it using formulas one could instead say:

$$u(x,0) = \begin{cases} 0 & \text{if } 0 \le x \le d-a \\ x+a-d & \text{if } d-a < x \le d \\ a+d-x & \text{if } d < x \le d+a \\ 0 & \text{if } x > a+d \end{cases} \tag{1}$$

You know the wave equation solution for this situation for $t > 0$ is $u(x,t) = \frac{1}{2}(u(x+ct,0) + u(x-ct,0))$ until $ct = x$, at which point it is $\frac{1}{2}(u(x+ct,0) - u(ct-x,0))$. In our case $x = 2d$, so for $t > 0$ we have $x + ct = 2d + ct > 2d > a + d$ and $u(x + ct, 0) = 0$. Thus we have

$$u(x,t) = \begin{cases} \frac{1}{2}u(x - ct, 0) & \text{if } 0 \leq ct \leq 2d \\ -\frac{1}{2}u(ct - x) & \text{if } 2d < ct < \infty \end{cases}$$

For $ct \leq 2d$ we must consider where $x - ct = 2d - ct$ is in the regions described in equation (1). The transitions are at $2d-ct = 0, d-a, d, d+a$ and so at $ct = 2d, d+a, d, d-a$. These give the first intervals in equation (2) below. Similarly, when $ct > 2d$ we have transitions at $ct - 2d = 0, d - a, d, d + a$, so at $ct = 2d, 3d - a, 3d, 3d + a$ which give the remaining intervals in (2)

$$u(2d,t) = \begin{cases} 0 & \text{if } 0 \leq ct \leq d - a \\ a - d + ct & \text{if } d - a < ct \leq d \\ a + d - ct & \text{if } d < ct \leq d + a \\ 0 & \text{if } d + a < ct \leq 3d - a \\ 3d - a - ct & \text{if } 3d - a < ct \leq 3d \\ -a - 3d + ct & \text{if } 3d < ct \leq 3d + a \\ 0 & \text{if } ct > 3d + a \end{cases} \tag{2}$$

Either way the picture is as follows

Math 3435: Formula sheet

Wave equation on $\mathbb{R}$

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

Wave equation on $(0,\infty)$ with Dirichlet boundary data if $0 < x < ct$

$$u(x,t) = \frac{1}{2}(\phi(x+ct) - \phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy$$

Wave equation on $(0,\infty)$ with Neumann boundary data if $0 < x < ct$

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(ct-x)) + \frac{1}{c}\int_{0}^{ct-x} \psi(y)dy + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy$$

Energy in wave equation

$$E(t) = \int u_t^2(y,t) + c^2 u_x^2(y,t)\,dy$$

Fundamental solution (or source) for heat equation if $t > 0$

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$$

Heat equation on $\mathbb{R}$ for $t > 0$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy$$

Heat equation on $(0,\infty)$ with Dirichlet boundary data for $t > 0$

$$u(x,t) = \int_0^{\infty} \Big(S(x-y,t) - S(x+y,t)\Big)\phi(y)\,dy$$

Heat equation on $(0,\infty)$ with Neumann boundary data for $t > 0$

$$u(x,t) = \int_0^{\infty} \Big(S(x-y,t) + S(x+y,t)\Big)\phi(y)\,dy$$

Energy in heat equation

$$E(t) = \int u^2(y,t)\,dy$$

Error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp$$